Weakening Path Contributions and Progressive Quantization in Iterated HD Systems

Consider the iterated normalized Hadamard-sign system

$$x_{t+1} = \hat{H} D x_t$$

where

$$\hat{H} = \frac{1}{\sqrt{N}} H$$

and:

- H is the Walsh-Hadamard matrix,
- D is a diagonal sign matrix with entries (±1),
- $N = 2^n$.

A useful viewpoint is to interpret the iterated system as a branching path expansion over the recursive butterfly structure of the Fast Walsh-Hadamard Transform (FWHT).

After k iterations, each output coordinate may be written as a sum over exponentially many signed computational paths through the recursive interaction graph. Each iteration introduces:

- new parity-conditioned mixing through H,
- new sign/XOR-like modulation through D,
- and further subdivision of amplitude contributions.

The important scaling effect comes from the normalization factor:

$$\frac{1}{\sqrt{N}}$$

which appears at every iteration level.

After k iterations, an individual path contribution has magnitude approximately

$$N^{-k/2}$$

while the number of contributing paths grows exponentially with depth.

Thus:

- path count increases rapidly,
- individual path magnitude decreases rapidly,
- total norm remains preserved.

This produces a progressively finer decomposition of the final output coordinates.

A useful interpretation is that increasing iteration depth generates increasingly fine angular or interference resolution on the sphere:

- early iterations produce coarse large-scale rotational structure,
- deeper iterations contribute smaller corrective interference components,
- progressively lower-amplitude contributions encode increasingly refined structure.

In the unnormalized transform, the outputs are sums of many signed inputs:

$$y_i = \sum_j \pm x_j$$

and therefore require additional arithmetic precision. For a width-N transform, approximately

$\log_2 N$

additional bits are required to exactly represent worst-case outputs.

For example:

- 8-bit inputs,
- width 256 transform,
- require roughly 16-bit outputs.

This arithmetic growth reflects coherent accumulation of many parity-conditioned contributions.

Under iteration, the normalized system may therefore be viewed as repeatedly:

1. generating exponentially many finer path contributions,
2. attenuating each individual contribution by repeated scaling,
3. preserving total norm globally,
4. and recursively refining the representable interference structure.

The diagonal sign matrix D plays a particularly important role because Walsh functions are themselves parity/XOR structures:

$$H_{a,b} = (-1)^{a \cdot b}$$

with binary inner product mod 2.

Thus each application of D:

- modifies parity alignment,
- changes interference relationships,
- and injects new XOR-like structure before the next global mixing stage.

From the recursive tree viewpoint:

- deep paths correspond to increasingly fine-scale parity interactions,
- while the scaling factor continuously pushes those effects into lower-amplitude components.

One may therefore interpret iteration depth as generating progressively finer quantization of the output state:

- coarse structure appears first,
- fine structure emerges later through exponentially many weak contributions,
- and the lower-significance arithmetic structure encodes the deepest parity-interference histories.

This gives iterated HD systems a hybrid character:

- combinatoric and digital from the XOR/parity structure,
- analog and geometric from norm-preserving rotational evolution,
- and multiscale from the recursive attenuation of path amplitudes with depth.